International Journal of Research in Advent Technology, Vol.5, No.9, September 2017
E-ISSN: 2321-9637
Available online at www.ijrat.org

Fekete- Szego Inequality for a new subclass of Starlike Function

Gaganpreet Kaur[1], Gurmeet Singh[2]
[1]*Department of Mathematics, Punjabi University, Patiala.*
[2] *Department of Mathematics, GSSDGS Khalsa College Patiala*
Email: kaurgaganpreet91@gmail.com[1], meetgur111@gmail.com[2]

Abstract- The aim of the present paper is to investigate a certain subclass of starlike function and obtain the sharp upper bound of the functional $\left|a_3-\mu a_2^2\right|$ for the analytic function

$$f(z)=z+a_2z^2+a_3z^3+\cdots,\qquad |z|<1$$

belonging to this subclass of starlike function.

2000 Mathematical Subject Classification: 30C45, 30C50.

Keywords: Analytic Function, Bounded function, Fekete Szego Inequality, Starlike Function, Subordination, Univalent Function.

1 INTRODUCTION

Let A denote the class of functions of the form

$$f(z)=z+a_2z^2+a_3z^3+\cdots \tag{1}$$

Which are analytic in the open unit disc $U=z:z\in C;|z|<1$ and let S denote the class of functions in A that are univalent in U.

In 1916, for the functions $f(z)\in S$, Bieber Bach [4, 5] proved the result $|a_2|\le 2$. In 1923, for the same functions, Lowner [2] proved that $|a_3|\le 3$. With these results $|a_2|\le 2$ and $|a_3|\le 3$, for the class S it was very easy to draw out the relation between a_3 and a_2^2. With the help of Lowner's method , Fekete and Szego [6] proved the following well known result

$$\left|a_3-\mu a_2^2\right|\le\begin{cases}3-4\mu & if \quad \mu\le 0\\ 1+\exp\left(\dfrac{-2\mu}{1-\mu}\right) & if \quad 0\le\mu\le 1\\ 4\mu-3 & if \quad \mu\ge 1\end{cases}$$

This inequality is very much helpful in determining estimates of higher coefficients for some subclasses S (See Chhichra [1], Babalola [3]).

Now we define some subclasses of S. Let

$$f(z)=z+a_2z^2+a_3z^3+\cdots\in A$$

and satisfy the condition

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right)>0,\quad z\in U$$

is univalent starlike function and denoted by

S^* and a subclass $S*(A,B)=\left\{ f(z)\in A, \frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1+Bz} \right.$ where $-1\le B<A\le 1, z\in U$

It is obvious that $S^*(A,B)$ is a subclass of S^*.

We introduce a new class as

$$S^*(p)=\left\{ f(z)\in A, \frac{zf'(pz)}{pf(z)} \prec \frac{1+z}{1-z}, z\in U \right\}$$

Symbol $\prec$ stands for subordination.

Analytic bounded functions: Class of analytic bounded function is of the form

$$w(z)=\sum_{n=1}^{\infty} c_n z^n, w(0)=0, |w(z)|\le 1.$$

It is known that $|c_1|\le 1, |c_2|\le 1-|c_1|^2$.

2 FEKETE-SZEGO PROBLEM

Our main result is the following

2.1 Theorem

Let the bounded function $w(z)=c_1z+c_2z^2+\cdots$ and $f(z)\in S^*(p)$, then

$$|a_2^2-\mu a_3|\le \begin{cases} \frac{2(2p+1)}{(3p^2-1)(2p-1)}-\frac{4}{(2p-1)^2}\mu & \text{if } \mu\le\lambda_1; \\ \frac{2}{3p^2-1} & \text{if } \lambda_1<\mu<\lambda_2 \\ \frac{4}{(2p-1)^2}-\frac{2(2p+1)}{(3p^2-1)(2p-1)} & \text{if } \mu\ge\lambda_2 \end{cases}$$

where $\lambda_1=\frac{2p-1}{3p^2-1}$ and $\lambda_2=\frac{2p(2p-1)}{3p^2-1}$

The results are sharp.

Proof: By definition of $S^*(p)$, we have

$$\frac{zf'(pz)}{pf(z)}=\frac{1+w(z)}{1-w(z)} \qquad \ldots(2)$$

By expending the series (2)

$$1+(2p-1)a_2z+\left((1-2p)a_2^2+(3p^2-1)a_3\right)z^2+\cdots = 1+2c_1z+2(c_2+c_1^2)z^2+\cdots \qquad \ldots(3)$$

Comparing coefficients of (3)

$$a_2=\frac{2c_1}{2p-1} \quad \text{and}$$

$$a_3=\frac{2}{(3p^2-1)}c_2+\frac{2(2p+1)}{(3p^2-1)(2p-1)}c_1^2 \qquad \ldots(4)$$

$$|a_3-\mu a_2^2|\le \frac{2}{(3p^2-1)}|c_2|+ \left\{\frac{2(2p+1)}{(3p^2-1)(2p-1)}-\frac{4}{(2p-1)^2}\mu\right\}|c_1^2|$$

$$=\frac{2}{3p^2-1}+ \left\{\left|\frac{2(2p+1)}{(3p^2-1)(2p-1)}-\frac{4}{(2p-1)^2}\mu\right|-\frac{2}{3p^2-1}\right\}|c_1|^2 \qquad \ldots(5)$$

Case 1: when $\mu\le\frac{4p^2-1}{2(3p^2-1)}$

Inequality (5) can be rewritten as

$$\left|a_3-\mu a_2^2\right| \le \frac{2}{\left(3p^2-1\right)}+\left\{\frac{2(2p+1)}{\left(3p^2-1\right)(2p-1)}-\frac{4}{(2p-1)^2}\mu-\frac{2}{3p^2-1}\right\}\left|c_1^2\right| \quad \ldots(6)$$

Sub case 1(a): When $\mu \le \frac{2p-1}{3p^2-1}$,

Then equation (6) can be rewritten as

$$\left|a_3-\mu a_2^2\right| \le \frac{2(2p+1)}{\left(3p^2-1\right)(2p-1)}-\frac{4}{(2p-1)^2}\mu \quad \ldots(7)$$

Sub case 1(b): When $\frac{2p-1}{3p^2-1} < \mu < \frac{4p^2-1}{2\left(3p^2-1\right)}$

then the equation (6) becomes

$$\left|a_3-\mu a_2^2\right| \le \frac{2}{\left(3p^2-1\right)}-\left\{\frac{2(2p+1)}{\left(3p^2-1\right)(2p-1)}-\frac{4}{(2p-1)^2}\mu-\frac{2}{3p^2-1}\right\}\left|c_1^2\right|$$

$$\left|a_3-\mu a_2^2\right| \le \frac{2}{\left(3p^2-1\right)} \quad \ldots(8)$$

Case 2: When $\mu \ge \frac{4p^2-1}{2\left(3p^2-1\right)}$,

$$\left|a_3-\mu a_2^2\right| \le \frac{2}{\left(3p^2-1\right)}+\left\{-\frac{2(2p+1)}{\left(3p^2-1\right)(2p-1)}+\frac{4}{(2p-1)^2}\mu-\frac{2}{3p^2-1}\right\}\left|c_1^2\right| \quad \ldots(9)$$

Sub case 2(a): When $\mu \ge \frac{2p(2p-1)}{3p^2-1}$,

Then the equation (9) becomes

$$\left|a_3-\mu a_2^2\right| \le -\frac{2(2p+1)}{\left(3p^2-1\right)(2p-1)}+\frac{4}{(2p-1)^2}\mu \quad \ldots(10)$$

Sub case 2(b): When

$\frac{4p^2-1}{2\left(3p^2-1\right)} < \mu < \frac{2p(2p-1)}{3p^2-1}$,

Then the equation (9) becomes

$$\left|a_3-\mu a_2^2\right| \le \frac{2}{\left(3p^2-1\right)}-\left\{-\frac{2(2p+1)}{\left(3p^2-1\right)(2p-1)}+\frac{4}{(2p-1)^2}\mu-\frac{2}{3p^2-1}\right\}\left|c_1^2\right|$$

$$\left|a_3-\mu a_2^2\right| \le \frac{2}{\left(3p^2-1\right)} \quad \ldots(11)$$

Combining the equations (7), (8), (10) and (11).

We get the Fekete Szego inequality for $S^*(p)$ as

$$\left|a_3-\mu a_2^2\right| \le \begin{cases} \dfrac{2(2p+1)}{(3p^2-1)(2p-1)}-\dfrac{4}{(2p-1)^2}\mu \\ \quad if \quad \mu \le \dfrac{2p-1}{3p^2-1}; \\ \dfrac{2}{3p^2-1} \\ \quad if \quad \dfrac{2p-1}{3p^2-1} < \mu < \dfrac{2p(2p-1)}{3p^2-1}; \\ \dfrac{4}{(2p-1)^2}\mu-\dfrac{2(2p+1)}{(3p^2-1)(2p-1)} \\ \quad if \quad \mu \ge \dfrac{2p(2p-1)}{3p^2-1} \end{cases}$$

The extremal function for first and third inequality is

$$f_1(z)=z\left\{1+\frac{2p^2z}{(1-3p^2)(2p-1)}\right\}^{\frac{1-3p^2}{p^2}}$$

And extremal function for second inequality is

$$f_2(z)=z\{1+2z^2\}^{\frac{1}{3p^2-1}}$$

Corollary: Putting $p=1$ in the theorem 2.1 we get

$$\left|a_3-\mu a_2^2\right| \le \begin{cases} 3-4\mu & if \quad \mu \le \frac{1}{2}; \\ 1 & if \quad \frac{1}{2} < \mu < 1; \\ 4\mu-3 & if \quad \mu \ge 1 \end{cases}$$

Which is the result obtained by [8].

REFERENCE

[1] P.N. Chichra, New subclasses of the class of close- to- convex functions, Procedure of American Mathematical Society, 62(1977), 37-43.
[2] K. Lwner, Uber monotone Matrixfunktion, Math. Z., 38(1934), 177-216.
[3] K.O. Babalola, The fifth and sixth coefficients of close-to-convex functions, Kragujevac J. Math., 32(2009), 5-12.
[4] L. Bieberbach, Uber einige extremal problem in Gebiete der konformen abbildung, Math., Ann., 77(1916), 153-172.
[5] L. Bieberbach, Uber die koeffizientem derjenigen potenzreihen, welche eine schlithe abbildung des einheitskreises vermitteln, Preuss. Akad. Wiss Sitzungsb.,138(1916), 940-955.
[6] M. Fekete and G. Szeg,8(1933): Eine bemerkung ber ungerade schlichte funktionen, J London Math. Soc., 85-89.
[7] Z. Nehari,(1952): Conformal Mapping, Mc-Graw-Hill, Comp. Inc., New York.